SYZYGIES, REGULARITY AND TORIC VARIETIES

MILENA HERING

ABSTRACT. Let $\mathcal{A}$ be an ample line bundle on a projective toric variety X of dimension n. We show that if $\ell \geq n-1+p$, then $\mathcal{A}^\ell$ satisfies the property N_p. Applying similar methods, we obtain a combinatorial theorem: For a given lattice polytope P we give a criterion for an integer m to guarantee that mP is normal.

1. INTRODUCTION

Let $\mathcal{L}$ be a globally generated, ample line bundle on a projective variety X over a field k of characteristic zero and let $\phi_{\mathcal{L}}$ be the map of X into $\mathbb{P}(H^0(X, \mathcal{L}))$. Let S be the symmetric algebra $\mathrm{Sym}^\bullet H^0(X, \mathcal{L})$ and let $R = \bigoplus_m H^0(X, \mathcal{L}^m)$, a finitely generated S-module.

Let

$$0 \to E_k \to \cdots \to E_1 \to E_0 \to R \to 0$$

be a minimal free graded resolution of the S-module R. E_i is also called the *i'th syzygy module* of R.

We say that an ample line bundle $\mathcal{L}$ *satisfies property* N_0 if $E_0 = S$, and it *satisfies property* N_p if $E_0 = S$ and $E_i \cong \bigoplus S(-i-1)$ for $1 \leq i \leq p$.

For example, if $\mathcal{L}$ satisfies N_0, $\mathcal{L}$ is very ample and if X is normal, $\phi_{\mathcal{L}}$ embeds X into $\mathbb{P}(H^0(X, \mathcal{L})$ as a projectively normal variety. If it also satisfies N_1, it has quadratic defining equations and N_2 implies that the relations among these equations are linear. For $p = 0, 1$, the property N_p has been studied by Mumford in [Mum70], who called such line bundles *normally generated* and *normally presented*, respectively. For a survey of this property, we refer to [Laz], Section 1.8.D.

Recall that for an ample and globally generated line bundle $\mathcal{A}$ on X, a sheaf $\mathcal{F}$ is called *m-regular with respect to* $\mathcal{A}$ in the sense of Castelnuovo-Mumford, if

$$H^i(X, \mathcal{F} \otimes \mathcal{A}^{m-i}) = 0 \text{ for all } i \geq 1.$$

Date: February 1, 2008.

Definition 1.1. We will call $\mathcal{A}$ *m-autoregular*, if it is m-regular with respect to itself, i.e., if

$$H^i(X, \mathcal{A}^{m+1-i}) = 0 \text{ for all } i \geq 1.$$

The following theorem is due to Gallego and Purnaprajna. We include a short proof in this paper.

Theorem 1.2 ([GP99], Theorem 1.3.)**.** *Let $\mathcal{A}$ be an ample line bundle on X that is globally generated, and suppose that $\mathcal{A}$ is m-autoregular for some integer m. If $\ell \geq \max\{1, m+p\}$ and $p \geq 1$, then $\mathcal{A}^\ell$ satisfies property N_p.*

This theorem has a particularly nice application to an ample line bundle $\mathcal{A}$ on a projective toric varietiy X of dimension $n \geq 2$. Every ample line bundle $\mathcal{A}$ on a toric variety of dimension n is $(n-1)$-autoregular, since it is globally generated and since the higher cohomology of a globally generated line bundle vanishes. Hence we can apply Theorem 1.2 to $\mathcal{A}$. Together with the fact that if $\ell \geq n-1$, then $\mathcal{A}^\ell$ satisfies property N_0 ([EW91], [LTZ93], [ON02]), we obtain the following Corollary.

Corollary 1.3. *Let $\mathcal{A}$ be an ample line bundle on a toric variety X of dimension $n \geq 2$. Then $\mathcal{A}^\ell$ satisfies property N_p when $\ell \geq n-1+p$ and $p \geq 0$.*

This result has first appeared in a preprint by Hal Schenck and Gregory Smith [SS03]. We can fix a gap in their proof by applying Theorem 1.2.

The proofs of the case $p = 0$ in [EW91], [LTZ93] and [ON02] use the combinatorics of lattice points in polytopes. Nakagawa and Ogata prove the case $p = 1$ following Mumford in [Mum70]; in [BGT97], Bruns, Gubeladze and Trung prove the case $p = 0$ and $p = 1$ in a special case using the commutative algebra associated to polytopal semigroup rings.

Koelman gives a combinatorial criterion for an ample line bundle on a toric surface to be normally presented in [Koe93]: Let $\mathcal{L} = \mathcal{O}_X(D)$ for some torus invariant Cartier divisor D and let P_D be the polytope associated to D. Recall that $P_{mD} = mP_D$. Then Koelman proves that $\mathcal{O}_X(D)$ is normally presented if P_D contains more than 3 lattice points in its boundary; in particular he shows that for any ample line bundle $\mathcal{A}$ on a toric surface, $\mathcal{A}^2$ is normally presented.

Investigating the regularity of ample line bundles on toric varieties, we obtain an interesting corollary for lattice polytopes. Recall that a lattice polytope is *normal* if every lattice point in mP is the sum

of m lattice points in P. Note that on a toric variety X, $\mathcal{O}_X(D)$ is normally generated if and only if P_D is normal. Moreover, let $d(P)$ be the largest integer such that $d(P)P$ does not contain any lattice points in its relative interior.

Corollary 1.4. *Let V be a real vector space of dimension n, and let $M \subset V$ be a lattice of full rank. Let P be a lattice polytope of dimension n. Then ℓP is a normal polytope for all $\ell \geq \max\{n - d(P), 1\}$.*

In section 2 we recall the cohomological criterion of Green, Ein and Lazarsfeld ([EL93]) for a line bundle to satisfy the property N_p, and some basic facts about the regularity of coherent sheaves. In section 3 we will use these facts to prove Theorem 1.2. In the last section we give a combinatorial condition for the regularity of ample line bundles on toric varieties, and in this way we obtain some better bounds than in Corollary 1.3.

I have learnt most of the techniques and theorems from the preprint by Hal Schenck and Gregory Smith [SS03]; in particular, the idea to use the regularity of powers of the vector bundles $M_{\mathcal{L}}$ defined in (1), which had also been applied by Gallego and Purnaprajna in [GP99]. Similar methods appear already in [EL93].

I wish to thank W. Fulton and R. Lazarsfeld for helpful discussions related to this work, and also A. Bayer for comments on earlier versions of this note.

2. Preliminaries

To a globally generated line bundle $\mathcal{L}$ we associate a vector bundle $M_{\mathcal{L}}$, defined by the following exact sequence:

$$0 \to M_{\mathcal{L}} \to H^0(X, \mathcal{L}) \otimes O_X \to \mathcal{L} \to 0. \tag{1}$$

The following cohomological criterion for the property N_p is the main tool in the proof of the theorem. It can be found in [EL93].

Theorem 2.1. *Let $\mathrm{char}(k) = 0$ and $\mathcal{L}$ be an ample line bundle that is globally generated. Then $\mathcal{L}$ satisfies property N_p if*

$$H^1(X, M_{\mathcal{L}}^{\otimes k} \otimes \mathcal{L}^j) = 0 \quad \text{for} \quad 0 \leq k \leq p+1 \quad \text{and} \quad j \geq 1.$$

Moreover, we will need the following properties of regular sheaves.

Remark 2.2. It follows from the definition that when $\mathcal{F}$ is m-regular with respect to $\mathcal{A}$, then $\mathcal{F} \otimes \mathcal{A}^d$ is $(m-d)$-regular with respect to $\mathcal{A}$ for all $d \in \mathbb{Z}$.

Lemma 2.3. *Let $\mathcal{A}$ be an ample and globally generated line bundle on X and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{2}$$

be a short exact sequences of coherent sheaves on X. Suppose that $\mathcal{F}$ is r-regular with respect to $\mathcal{A}$, that $\mathcal{F}''$ is $(r-1)$-regular with respect to $\mathcal{A}$, and that the map of global sections

$$H^0(X, \mathcal{F} \otimes \mathcal{A}^{r-1}) \to H^0(X, \mathcal{F}'' \otimes \mathcal{A}^{r-1}) \tag{3}$$

is surjective. Then $\mathcal{F}'$ is r-regular with respect to $\mathcal{A}$.

This follows from twisting (2) with appropriate powers of $\mathcal{A}$ and applying the definition of regularity to the long exact sequence of cohomology groups.

Theorem 2.4 (Mumford's Theorem, [Laz] Theorem 1.8.5.)**.** *Let $\mathcal{A}$ be an ample and globally generated line bundle on a projective variety X. Let $\mathcal{F}$ be a sheaf on X that is m-regular with respect to $\mathcal{A}$. Then for every $\ell \geq 0$:*

(1) $\mathcal{F} \otimes \mathcal{A}^{m+\ell}$ is generated by its global sections.
(2) The natural maps

$$H^0(X, \mathcal{F} \otimes \mathcal{A}^m) \otimes H^0(X, \mathcal{A}^\ell) \to H^0(X, \mathcal{F} \otimes \mathcal{A}^{m+\ell})$$

are surjective.
(3) $\mathcal{F}$ is $(m+\ell)$-regular with respect to $\mathcal{A}$.

3. The relation between regularity and the property N_p

Proof of Theorem 1.2. Let $\mathcal{L} = \mathcal{A}^\ell$ with $\ell \geq \max\{1, m+1\}$. Since $\mathcal{L}$ is ample and globally generated, we can associate a vector bundle $M_\mathcal{L}$ to it as in (1).

We claim that $M_\mathcal{L}^{\otimes k}$ is $(m+k)$-regular with respect to $\mathcal{A}$ for $k \geq 1$. Granting this claim, we see that in particular, $H^1(X, M_\mathcal{L}^{\otimes k} \otimes \mathcal{A}^d) = 0$ for $d \geq m+k-1$. Therefore, when $\ell \geq \max\{m+p, 1\}$ and $j \geq 1$, $j\ell \geq \ell \geq m+p \geq m+k-1$ for $p+1 \geq k \geq 0$ and hence $H^1(X, M_\mathcal{L}^{\otimes k} \otimes \mathcal{L}^j) = 0$ in this case. When $k=0$, the vanishing follows from the m-regularity of $\mathcal{A}$. It follows from Theorem 2.1 that $\mathcal{L}$ satisfies the property N_p.

We will prove the claim by induction on k, applying Lemma 2.3 with $r = m+k$ to the short exact sequence (1) defining $M_\mathcal{L}$ twisted by $M_\mathcal{L}^{\otimes(k-1)}$.

The case $k=1$ is a special case of Lemma 3.1. of [AK02], but we can see it directly as follows. Observe that since $\ell - 1 \geq m$, $\mathcal{A}$ is $(\ell-1)$-autoregular and so we can apply Mumford's theorem (2) to $\mathcal{A}$ to see

that for $\mathcal{L} = \mathcal{A} \otimes \mathcal{A}^{\ell-1}$, the map of global sections

$$H^0(X,\mathcal{L}) \otimes H^0(X,\mathcal{A}^m) \to H^0(X,\mathcal{L} \otimes \mathcal{A}^m)$$

is surjective. Using Remark 2.2 and the fact that cohomology commutes with tensoring with a vector space, we see that $H^0(X,\mathcal{L}) \otimes \mathcal{O}_X$ is $(m+1)$-regular with respect to $\mathcal{A}$. Similarly, $\mathcal{L}$ is $(m-\ell+1)$-regular, hence m-regular with respect to $\mathcal{A}$ since $\ell \geq 1$. Now Lemma 2.3 implies that $M_{\mathcal{L}}$ is $(m+1)$-regular with respect to $\mathcal{A}$.

For $k > 1$, we substitute $\mathcal{A}^\ell$ for $\mathcal{L}$ in (3) and apply (2) in Mumford's theorem to $M_{\mathcal{L}}^{\otimes(k-1)}$, which is $(m+k-1)$-regular by the induction hypothesis, to see that the map of global sections

$$\begin{aligned} H^0(X, M_{\mathcal{L}}^{\otimes(k-1)} \otimes \mathcal{A}^{m+k-1}) \otimes H^0(X,\mathcal{A}^\ell) \\ \to H^0(X, M_{\mathcal{L}}^{\otimes(k-1)} \otimes \mathcal{A}^{m+k-1+\ell}) \end{aligned}$$

is surjective. Moreover, by the induction hypothesis $M_{\mathcal{L}}^{\otimes(k-1)} \otimes H^0(X,\mathcal{L})$ is $(m+k-1)$-regular, in particular it is $(m+k)$-regular; similarly $M_{\mathcal{L}}^{\otimes(k-1)} \otimes \mathcal{L}$ is $(m+k-1)$-regular, since $\ell \geq 1$. Now the claim follows from Lemma 2.3. $\square$

Remark 3.1. That $\mathcal{A}^{m+1}$ satisfies property N_0 follows directly from Mumford's theorem (2).

4. The regularity of ample line bundles on toric varieties

In this section we will use a theorem by David Cox and Alicia Dickenstein to compute the regularity of a given line bundle on a toric variety in terms of combinatorial properties of a polytope associated to the line bundle.

Theorem 4.1 ([CD03], Theorem 1.3.)**.** *Let D be a torus invariant Cartier divisor on a complete toric variety X and let P_D be the polytope associated to D. Assume that $\mathcal{O}_X(D)$ is globally generated. Then*

(1) $H^i(X,\mathcal{O}_X(-D)) = 0$ for all $i \neq \dim(P_D)$.
(2) There is an isomorphism

$$H^{\dim(P_D)}(X,\mathcal{O}_X(-D)) \cong \bigoplus_{u \in \mathrm{relint}(P_D)\cap M} \mathbb{C}\chi^{-u}$$

that is equivariant with respect to the torus action.

In particular, if P_D has no interior lattice points, the cohomology of $\mathcal{O}_X(-D)$ vanishes. This motivates the definition of $d(P)$ for a polytope P in the introduction.

Definition 4.2. Let $\mathcal{A}$ be an ample line bundle that is globally generated. We define the *autoregularity* of $\mathcal{A}$ to be the smallest intger m such that $\mathcal{A}$ is m-autoregular.

Proposition 4.3. *Let D be a torus invariant ample divisor on a complete toric variety X of dimension n and let P_D be the polytope associated to D. Let $m = n-1-d(P_D)$. Then the autoregularity of $\mathcal{O}_X(D)$ is m.*

Proof. We have to show that, for $i \geq 1$,

$$H^i(X, \mathcal{O}_X((m+1-i)D)) = 0.$$

First recall that on a toric variety for any ample divisor D, $\mathcal{O}_X(D)$ is globally generated. So when $m+1-i \geq 0$, the statement follows from the vanishing of the higher cohomology of globally generated line bundles on any complete toric variety. (See for example [Ful93], section 3.4.).

When $m+1-i < 0$, $-(m+1-i)D$ is a positive integer multiple of D, in particular it is globally generated and we can apply Theorem 4.1. Since the dimension of the polytope associated to an ample divisor equals the dimension of the variety, the assertion follows for $i \neq n$. When $i = n$, $m+1-n = -d(P_D)$. But $d(P_D)P_D$ does not contain any interior lattice points by definition, and so Theorem 4.1 (2) implies that also $H^n(X, \mathcal{O}_X(-d(P_D)D) = 0$. Moreover $H^n(X, \mathcal{O}_X(-(d(P_D)+1)D)) \neq 0$, since $(d(P_D)+1)P$ does contain interior lattice points, so $\mathcal{O}_X(D)$ is not $(m-1)$-autoregular. □

Applying Theorem 1.2 to an ample divisor D on a toric variety X, we obtain the following Corollary, which implies Corollary 1.4.

Corollary 4.4. *Let D be a torus invariant ample divisor on a complete toric variety X of dimension n, let P_D be the polytope associated to D and let $m = n-1-d(P_D)$. If $p \geq 1$ and $\ell \geq \max\{m+p, 1\}$, then the line bundle $\mathcal{O}_X(\ell D)$ satisfies property N_p. In particular $(m+1)D$ is very ample and normally generated.*

References

[AK02] Donu Arapura and Dennis Keeler. Frobenius amplitude and strong vanishing theorems for vector bundle. 2002. math.AG/0202129.

[BGT97] Winfried Bruns, Joseph Gubeladze, and Ngô Viêt Trung. Normal polytopes, triangulations, and Koszul algebras. *J. Reine Angew. Math.*, 485:123–160, 1997.

[CD03] David Cox and Alicia Dickenstein. Vanishing and codimension theorems for complete toric varieties. 2003. math.AG/0310108v1.

[EL93] Lawrence Ein and Robert Lazarsfeld. Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension. *Invent. Math.*, 111(1):51–67, 1993.

[EW91] Günter Ewald and Uwe Wessels. On the ampleness of invertible sheaves in complete projective toric varieties. *Results Math.*, 19(3-4):275–278, 1991.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.

[GP99] F.J. Gallego and B.P. Purnaprajna. Projective normality and syzygies of algebraic surfaces. *J. Reine Angew. Math.*, 506:145–180, 1999.

[Koe93] Robert Jan Koelman. A criterion for the ideal of a projectively embedded toric surface to be generated by quadrics. *Beiträge Algebra Geom.*, 34(1):57–62, 1993.

[Laz] Robert Lazarsfeld. *Positivity in Algebraic Geometry*. to appear.

[LTZ93] Ji Yong Liu, Leslie E. Trotter, Jr., and Günter M. Ziegler. On the height of the minimal hilbert basis. *Results in Mathematics*, 23(3-4):374–376, 1993.

[Mum70] David Mumford. Varieties defined by quadratic equations. In *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, pages 29–100. Edizioni Cremonese, Rome, 1970.

[ON02] Shoetsu Ogata and Katsuyoshi Nakagawa. On generators of ideals defining projective toric varieties. *Manuscripta Math.*, 108(1):33–42, 2002.

[SS03] Hal Schenck and Gregory G. Smith. Syzygies of projective toric varieties. 2003. math.AG/0308205.